Journal of Computational and Applied Mathematics 35 (1991) 297–301
North-Holland

Uniform asymptotic expansions of a class of integrals with finite endpoints of integration on the same path of steepest descent and with nearby saddle points

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Received 30 May 1990

Abstract

Steinacker, U., C. Leubner and S.L. Kalla, Uniform asymptotic expansions of a class of integrals with finite endpoints of integration on the same path of steepest descent and with nearby saddle points, Journal of Computational and Applied Mathematics 35 (1991) 297–301.

For the frequently required uniform asymptotic expansion of a certain class of integrals that have both their finite endpoints on the same path of steepest descent with a number of nearby saddle points, a modification of an earlier procedure is given. Over Bleistein's general method it offers the convenience of completely separating the processes of determining the expansion coefficients and of repeatedly integrating by parts. This is of practical relevance with a view to delegating the otherwise very tedious calculation of higher expansion coefficients to one of the available computer codes for the algebraic-analytic manipulation of given expressions.

Keywords: Uniform asymptotic expansions, complex contour integration, method of steepest descent.

In physical applications, intermediate results are often obtained in terms of an integral over a closed-form integrand, with the integral itself not expressible in closed form. Usually, the integral depends on a number of parameters, and its value is required for a whole domain of the parameter space. In these situations, rather than producing a numerical table for isolated parameter combinations, it is in general more profitable to derive an asymptotic expansion, valid throughout certain domains of the parameter space.

One class of integrals, which frequently occurs in applications and for which closed-form approximations can be derived in a systematic manner, is of the form

$$I(\nu, \boldsymbol{p}) = \int_C A(t, \boldsymbol{p}) \exp[\mathrm{i}\nu\Phi(t, \boldsymbol{p})] \, \mathrm{d}t, \tag{1}$$

where C is a contour along the real axis, and $\boldsymbol{p}$ is a set of parameters. If the real part of $\Phi(t, \boldsymbol{p})$ varies along the given contour, then, for large real values of ν, the phase factor $\exp[\mathrm{i}\nu \operatorname{Re}\{\Phi(t, \boldsymbol{p})\}]$ causes very rapid oscillations of the integrand. Since this renders transparent approximations difficult in general, it is expedient to suppress these oscillations by deforming the contour of integration C by Cauchy's integral theorem into a new contour C_t, which is specified by the condition that $\operatorname{Re}\{\Phi(t, \boldsymbol{p})\}$ be constant along it.

Usually the deformed integration contour splits into several branches that have to be dealt with separately. The appropriate treatment of these branches depends on the type and on the distribution of neighboring critical points. Many such cases have already been considered in the literature [4].

In one situation that has not yet been considered but that nevertheless occurs in important applications, the integral in question is to be taken along a finite contour that starts at an endpoint and extends along a path of steepest descent to a saddle point of $\Phi(t, \boldsymbol{p})$, where it ends, with other saddle points nearby. It is the derivation of a uniform asymptotic expansion for this case on which we focus in this note, remarking also, however, on how to proceed in slightly more general cases as we go along.

It is useful to expound the key steps by way of an illustrative example. As such, we consider a class of incomplete cylindrical functions, which are widely used in physics and engineering [1,2,5,7],

$$\mathscr{E}_\nu(\omega, p) = -\frac{1}{\pi}\int_0^\omega \exp[\mathrm{i}\nu(t - p\sin t)]\,\mathrm{d}t, \quad \nu, \omega, p \text{ real}, \tag{2}$$

of which it is easy to show that the parameter space may be restricted to positive values of ν, ω and p, and, furthermore, to $\omega \leqslant \pi$. This function is related to the "Incomplete Cylindrical Function of Bessel form" $\epsilon_\nu(w, z)$, as defined in [1], through

$$\mathscr{E}_\nu(\omega, p) = \epsilon_\nu(-\mathrm{i}\omega, p\nu). \tag{3}$$

Obviously, $(-\pi)\mathscr{E}_\nu(\omega, p)$ is of the form (1), with

$$A(t, \boldsymbol{p}) = 1, \qquad \Phi(t, \boldsymbol{p}) = t - p\sin t. \tag{4}$$

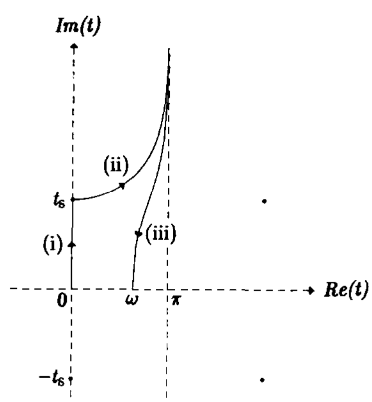

Fig. 1. "Nonoscillatory" contour of the integral (2) for $p = 0.1$, $\omega = 2$.

If $p < 1$, the original contour along the real axis (from 0 to ω) can be deformed into three "nonoscillatory" branches, denoted by (i), (ii) and (iii) in Fig. 1, with (i) being of the kind we are currently interested in. Throughout the rest of this paper, we denote the contribution from this branch to the total integral $\mathscr{E}_\nu(\omega, p)$ (which will be subjected to a closer investigation in a subsequent paper) by $\mathscr{E}_\nu(\omega, p)_{(\mathrm{i})}$,

$$\mathscr{E}_\nu(\omega, p)_{(\mathrm{i})} = -\frac{1}{\pi}\int_0^{t_\mathrm{S}} \exp[\mathrm{i}\nu(t - p \sin t)]\, \mathrm{d}t, \quad t_\mathrm{S} = \mathrm{i}\cosh^{-1}\left(\frac{1}{p}\right). \tag{5}$$

In principle, Bleistein's [3] method could be applied to (5), in which, as a first step, the integral is transformed to the so-called canonical form by means of a mapping

$$\Phi(t, \boldsymbol{p}) = P(z, z_\mathrm{S}^{(k)}), \tag{6}$$

where $P(z, z_\mathrm{S}^{(k)})$ is such a polynomial that it reproduces the configuration of those saddle points $t_\mathrm{S}^{(k)}$, $k = 1, 2, \ldots,$ of $\Phi(t, \boldsymbol{p})$ that affect the value of the given integral. In (5), the integrand assumes its maximum value at the lower endpoint of integration, and a pair of saddle points, t_S and $t_\mathrm{S}^* = -t_\mathrm{S}$ (cf. Fig. 1), may indefinitely approach this endpoint as p tends to 1. P is therefore suitably chosen as a polynomial of degree three,

$$P_3(z, z_\mathrm{S}) = \tfrac{1}{3}z^3 - z_\mathrm{S}^2 z. \tag{7}$$

With (4) and (7), (6) reads

$$t - p \sin t = \tfrac{1}{3}z^3 - z_\mathrm{S}^2 z, \tag{8}$$

and we are interested in that branch of this mapping which maps $t = 0$ onto $z = 0$, while the value of z_S is fixed by requiring that the original saddle points $\pm t_\mathrm{S}$ be mapped onto the saddle points $\pm z_\mathrm{S}$ of P_3.

Under the mapping (8), the integral (5) becomes

$$\mathscr{E}_\nu(\omega, p)_{(\mathrm{i})} = -\frac{1}{\pi}\int_0^{z_\mathrm{S}} \frac{\mathrm{d}t}{\mathrm{d}z} \exp[\mathrm{i}\nu P_3]\, \mathrm{d}z. \tag{9}$$

In a second step, we deviate from [3] by expanding $\mathrm{d}t/\mathrm{d}z$ in (9) into a series around $z = 0$ (where the integrand assumes its largest value) of the particular form

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \sum_{n=0}^{\infty} [A_n + B_n z + C_n z^2]\left[z\frac{\mathrm{d}P_3}{\mathrm{d}z}\right]^n. \tag{10}$$

The point of this expansion is that the factor $[z\, \mathrm{d}P_3/\mathrm{d}z]^n$ in the nth term allows us to perform, after insertion of (10) into (9), a certain number of successive integrations by parts (each of them yielding a factor $(\mathrm{i}\nu)^{-1}$), where the integrated term vanishes at either end of the contour of integration. As a consequence, it is found the nth term of (10) contributes at most to order $\mathrm{O}(\nu^{-\mathrm{Int}((n+1)/2)})$ to the resulting asymptotic series.

Let us remark here that in the slightly more general case of two finite endpoints of integration on a path of steepest descent, none of which is a saddle point, but with m relevant saddle points nearby, instead of (9) we would have

$$I(\nu, \boldsymbol{p}) = \int_{z_\mathrm{l}}^{z_\mathrm{u}} A(t, \boldsymbol{p}) \frac{\mathrm{d}t}{\mathrm{d}z} \exp[\mathrm{i}\nu P_{m+1}]\, \mathrm{d}z. \tag{11}$$

In this case, we would employ the expansion,

$$A(t,\,p)\frac{\mathrm{d}t}{\mathrm{d}z}=\sum_{n=0}^{\infty}\left[C_n^{(0)}+C_n^{(1)}(z-z_1)+C_n^{(2)}(z-z_1)^2+\cdots+C_n^{(m+1)}(z-z_1)^{m+1}\right]$$
$$\times\left[(z-z_u)(z-z_1)\frac{\mathrm{d}P_{m+1}}{\mathrm{d}z}\right]^n, \tag{12}$$

assuming that the integrand is maximal at the lower endpoint z_1. Here, the factor $[(z-z_u)(z-z_1)\,\mathrm{d}P_{m+1}/\mathrm{d}z]^n$ allows us to perform a certain number of successive integrations by parts, where the integrated term vanishes again at either end of the contour of integration.

Returning to (10), we note that whenever z^2 occurs, it may be replaced by $z_S^2+\mathrm{d}P_3/\mathrm{d}z$. That is why, after carrying out all possible integrations by parts up to a certain order $O(\nu^{-m})$, the resulting series involves only two different integrals, namely, the canonical integrals for the problem in hand,

$$\Gamma_n=\int_0^{z_S} z^n \exp[\mathrm{i}\nu P_3]\,\mathrm{d}z, \quad n=0,1. \tag{13}$$

By noting that

$$\left.\frac{\mathrm{d}t}{\mathrm{d}z}\right|_{\zeta}=\left.\frac{\mathrm{d}t}{\mathrm{d}z}\right|_{-\zeta}, \tag{14}$$

the coefficients in (10) may be simplified at the outset to

$$B_k=0, \quad \text{if } k \text{ is even}, \qquad A_k=0,\; C_k=0, \quad \text{if } k \text{ is odd}. \tag{15}$$

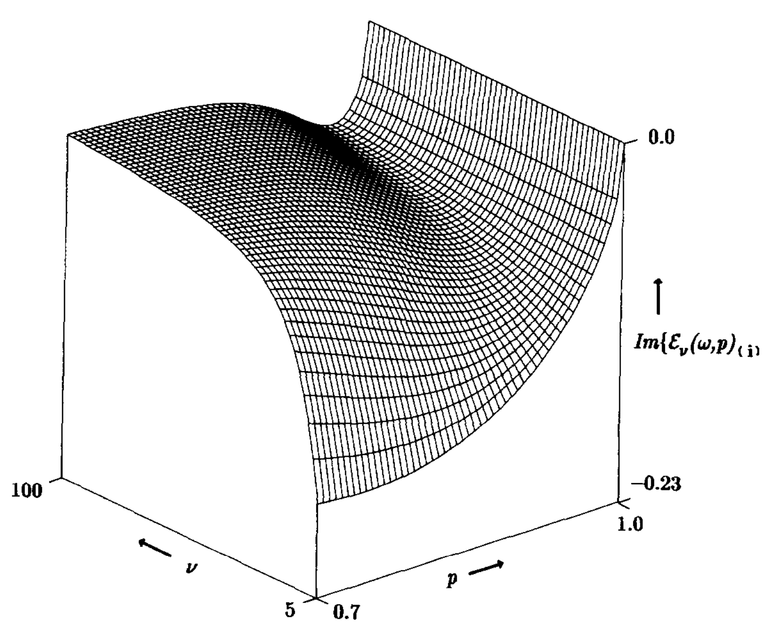

Fig. 2. Plot of the integral (5) and (16) for ranges of p and ν.

Explicitly, we find up to order $O(\nu^{-1})$ by the method described

$$\mathscr{E}_\nu(\omega,\, p)_{(\mathrm{i})} = -\frac{1}{\pi}\left\{(A_0 + C_0 z_S^2)\Gamma_0 - \frac{2}{\mathrm{i}\nu}(B_1 + A_2 z_S^2 + C_2 z_S^4)\Gamma_1 - \frac{C_0}{\mathrm{i}\nu} + \frac{\exp\left[-\mathrm{i}\left(\frac{2}{3}\right)\nu z_S^3\right]}{\mathrm{i}\nu}(C_0 + B_1 z_S^2)\right\} + O(\nu^{-2}). \tag{16}$$

Thus, the expansion (10) (or (12) for the more general case), which is a suitable modification of the method put forward in [6], indeed completely separates the processes of repeatedly integrating by parts and of determining the expansion coefficients. It therefore only remains to determine the latter for the particular example (5). As usual, A_n, B_n, C_n are found by differentiating (10), the left-hand side of which must be found from (8), $3(n+1)$ times with respect to z, and then setting $z = 0$.

It is well known that this algorithm, although simple, becomes excessively tedious for anything but the lowest coefficients. However, the advantage of the present method is that the calculation of the coefficients can be easily relegated to one of the available packages for algebraic-analytic computation, such as REDUCE, which was also used in the present case. With coefficients thus calculated, the result (16) — which is purely imaginary — is plotted in Fig. 2 for a domain of the parameter space that includes the coalescing of the two relevant saddle points.

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